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Leighton's Bounds for Sturm–Liouville Eigenvalues with Eigenvalue Parameter in the Boundary Conditions

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1. INTRODUCTION

In [3] (see also [12, 7, 2, 1]) an algorithm is described for computing upper and lower bounds for the eigenvalues of a regular Sturm–Liouville problem on a finite interval. The method replaces the coefficients of the differential equation by constants on each subinterval of a partition of the interval. The bounds converge to the true eigenvalues, the rate of convergence being of the first order in h (h being the step size). In [13] it is shown that the arithmetic mean of the upper and lower bounds exhibits second-order convergence.

In this paper we generalize the method to regular Sturm–Liouville problems on a finite interval with boundary conditions containing the eigenvalue parameter linearly. We prove the first-order convergence of the upper and lower bounds to the true eigenvalues and the second-order convergence of the arithmetic mean of the upper and lower bounds. For the proofs we use an operator-theoretic formulation of the problem which can be made for coefficients having jump discontinuities analogous to that given in [17, 8, 11, 4, 16] for continuous coefficients, and an error characterization similar to that used in [10] for some special singular eigenvalue problems with eigenvalue parameter in one boundary condition and to that used in [13] by the proof of the second-order convergence of the arithmetic means in the case of λ-independent boundary conditions.

2. NOTATION, THEORY, AND THE ALGORITHM

Let $[a, b] \subset \mathbb{R}$ a finite interval and let $\pi := \{a = x_0 < x_1 < \cdots < x_n = b\}$ a partition of $[a, b]$. For $m = 0, 1, 2, \ldots$ let $PC^m_\pi[a, b]$ denote the set of functions $f\colon [a, b] \to \mathbb{R}$ in

$$C^m[a, x_1) \cap C^m(x_1, x_2) \cap \cdots \cap C^m(x_{n-1}, b]$$

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such that $f^{(m)}$ has left and right limits at $x_1, \ldots, x_{n-1}$. Let $r \in PC_\pi^1[a, b]$, $p, q \in PC_\pi^0[a,b]$, where $r \geqslant r_0 > 0$, $p \geqslant p_0 > 0$ on $[a, b]$ with constants r_0, p_0. Let $\alpha_1, \alpha_2, \beta_1, \beta_1', \beta_2, \beta_2' \in \mathbb{R}$ with

$$|\alpha_1| + |\alpha_2| > 0 \qquad \text{and} \qquad \rho := \beta_1'\beta_2 - \beta_1\beta_2' > 0. \tag{1}$$

Then we consider the Sturm–Liouville eigenvalue problem with eigenvalue parameter in the boundary condition

$$(ry')' + (\lambda p - q)\, y = 0 \qquad \text{on} \quad (x_{i-1}, x_i),\ i = 1, \ldots, n, \tag{2}$$

$$\alpha_1 y(a) - \alpha_2 (ry')(a) = 0, \tag{3}$$

$$(\beta_1 + \beta_1'\lambda)\, y(b) - (\beta_2 + \beta_2'\lambda)(ry')(b) = 0, \tag{4}$$

y is absolutely continuous on $[a, b]$,
ry' is absolutely continuous on $[a, x_1)$, (x_{i-1}, x_i), $(x_{n-1}, b]$ with $\lim_{x \to x_i -} (ry')(x) = \lim_{x \to x_i +} (ry')(x)$. (5)

Some other boundary conditions can be considered similarly: The case of a λ-dependent boundary condition like (4) at $x = a$ and a λ-independent boundary condition like (3) at $x = b$ and the case of λ-dependent boundary conditions like (4) at both endpoints $x = a$ and $x = b$. In the two cases likewise a modified condition (1) must be fulfilled. For brevity we treat only the case of the boundary conditions (3), (4).

THEOREM 1. *For the eigenvalue problem* (2)–(5) *assume that* (1) *is fulfilled; then the eigenvalue problem* (2)–(5) *has a countable infinity of real* (*and simple*) *eigenvalues* $\lambda_1 < \lambda_2 < \ldots$ *and* $\lambda_k \to \infty$ *for* $k \to \infty$. *Let* y_k *denote the eigenfunction corresponding to the eigenvalue* λ_k. *If* $\beta_2' = 0$ *then* y_k *has exactly* $(k-1)$ *zeros in* (a, b). *If* $\beta_2' \neq 0$ *then the eigenfunctions* y_k *corresponding to eigenvalues* $\lambda_k < -\beta_2/\beta_2'$ *have exactly* $(k-1)$ *zeros in* (a, b) *and the eigenfunctions* y_k *corresponding to eigenvalues* $\lambda_k \geqslant -\beta_2/\beta_2'$ *have exactly* $(k-2)$ *zeros in* (a, b).

Proof. This can be proved analogous to the corresponding result in [4] for the Liouville normal form. ∎

We now describe the operator-theoretic formulation of the eigenvalue problem (2)–(5) where (1) is satisfied (see [4, 8, 11, 16, 17]). We define the Hilbert space H of two-component vectors by $H := L^2(p\colon a, b) \oplus \mathbb{C}$ with inner product given by

$$[(f_1, f_2), (g_1, g_2)] := \int_a^b p f_1 \overline{g_1} + \frac{1}{\rho} f_2 \overline{g_2}.$$

Let $D(A)$ be the set of all $(f_1, f_2) \in H$ which satisfy

f_1 is absolutely continuous on $[a, b]$,
rf'_1 is absolutely continuous on $[a, x_1)$, (x_{i-1}, x_i), $(x_{n-1}, b]$ with $\lim_{x \to x_i -} (rf'_1)(x) = \lim_{x \to x_i +} (rf'_1)$,
$-(rf'_1)' + qf_1 \in L^2(p; a, b)$, (6)

$$\alpha_1 f_1(a) - \alpha_2 (rf'_1)(a) = 0, \tag{7}$$

$$f_2 = \beta'_1 f_1(b) - \beta'_2 (rf'_1)(b) \tag{8}$$

and define $A: D(A) \to H$ by

$$A(f_1, f_2) = \left(\frac{1}{p} (-(rf'_1)' + qf_1), \ -\beta_1 f_1(b) + \beta_2 (rf'_1)(b) \right).$$

A is a self-adjoint operator in H. The eigenvalues of A are the eigenvalues of (2)–(5).

Let $D(J)$ be the set of all $(f_1, f_2) \in H$ which satisfy

$$f_1 \text{ is absolutely continuous with } f'_1 \in L^2(p; a, b), \tag{9}$$

$$f_1(a) = 0 \qquad \text{if} \quad \alpha_2 = 0 \text{ in (3)}, \tag{10}$$

if $\beta'_2 = 0$, then $f_2 = \beta'_1 f_1(b)$, and if $\beta'_2 \neq 0$, then f_1 is differentiable at b and $f_2 = \beta'_1 f_1(b) - \beta'_2 (rf'_1)(b)$. (11)

The function J is defined on $D(J) \times D(J)$ by

$\alpha_2 = 0, \qquad \beta'_2 = 0:$

$$J((f_1, f_2), (g_1, g_2)) := \int_a^b (rf'_1 \overline{g'_1} + qf_1 \overline{g_1}) - \frac{1}{\rho} \beta_1 \beta'_1 f_1(b) \overline{g_1(b)},$$

$\alpha_2 \neq 0, \qquad \beta'_2 = 0:$

$$J((f_1, f_2), (g_1, g_2)) := \int_a^b (rf'_1 \overline{g'_1} + qf_1 \overline{g_1}) - \frac{1}{\rho} \beta_1 \beta'_1 f_1(b) \overline{g_1(b)} + \frac{\alpha_1}{\alpha_2} f_1(a) \overline{g_1(a)},$$

$\alpha_2 \neq 0, \qquad \beta'_2 \neq 0:$

$$\begin{aligned} J((f_1, f_2), (g_1, g_2)) := \int_a^b (rf'_1 \overline{g'_1} + qf_1 \overline{g_1}) - \frac{1}{\rho} (\beta_1 \beta'_1 f_1(b) \overline{g_1(b)} \\ - \beta_1 \beta'_2 f_1(b) \overline{(rg'_1)(b)} - \beta_1 \beta'_2 (rf'_1)(b) \overline{g_1(b)} \\ + \beta_2 \beta'_2 (rf'_1)(b) \overline{(rg'_1)(b)}) + \frac{\alpha_1}{\alpha_2} f_1(a) \overline{g_1(a)}. \end{aligned}$$

If $(g_1, g_2) \in D(A)$, then

$$J((f_1, f_2), (g_2, g_2)) = [(f_1, f_2), A(g_1, g_2)], \qquad (f_1, f_2) \in D(J).$$

THEOREM 2. *For* $j = 1, 2$ *let* $r_j \in PC^1_\pi[a, b]$, $p_j, q_j \in PC^0_\pi[a, b]$, $r_1 \geqslant r_2 \geqslant r_0$ (const) > 0, $p_2 \geqslant p_1 \geqslant p_0$ (const) > 0, *and* $q_1 \geqslant q_2$ *on* (x_{i-1}, x_i), $i = 1, ..., n$. *Let* $(2^{(j)}), ..., (5^{(j)})$ *denote the eigenvalue problem* (2), ..., (5) *with* r, p, q *replaced by* r_j, p_j, q_j *and the corresponding eigenvalues by* $\lambda_1^{(j)} < \lambda_2^{(j)} < ...$, $j = 1, 2$. *Then*

$$\lambda_k^{(1)} \geqslant \lambda_k^{(2)} \tag{12}$$

for each $k \in \mathbb{N}$ *for which*

$$\lambda_k^{(1)}(p_1 - p_2) \leqslant 0. \tag{13}$$

Proof. First we follow the lines of the proof of Theorem 2.2(b) in [3]. But the appearance of the eigenvalue parameter in the boundary condition makes the proof somewhat more complicated. We must use our Theorem 1 and Theorem 2.1 in [3]: Suppose that

$$\lambda_k^{(1)} < \lambda_k^{(2)}. \tag{14}$$

Then it follows that

$$\begin{aligned}(\lambda_k^{(1)} p_1 - q_1) &- (\lambda_k^{(2)} p_2 - q_2)\\ &= (\lambda_k^{(1)} - \lambda_k^{(2)})\, p_2 + \lambda_k^{(1)}(p_1 - p_2) + q_2 - q_1 < 0.\end{aligned}$$

Case I. $\beta_2' = 0$: Then Theorem 1 implies that $y_k^{(1)}$ and $y_k^{(2)}$ have exactly $(k-1)$ zeros in (x_0, x_n), respectively. Furthermore Theorem 2.1 in [3] implies that $y_k^{(1)}(x_n) \neq 0$. The assumption $y_k^{(2)}(x_n) = 0$ contradicts condition (1). The assumption $y_k^{(2)}(x_n) \neq 0$ implies by again using Theorem 2.1 in [3] that $(r_1(y_k^{(1)})'/y_k^{(1)})(x_n) > (r_2(y_k^{(2)})'/y_k^{(2)})(x_n)$. With (5) the latter is equivalent to

$$\frac{\beta_1 + \beta_1' \lambda_k^{(1)}}{\beta_2} > \frac{\beta_1 + \beta_1' \lambda_k^{(2)}}{\beta_2}.$$

This contradicts (14) and the strictly increasing of the function $\beta\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\beta(\lambda) := \frac{\beta_1 + \beta_1' \lambda}{\beta_2}$$

(the strictly increasing of β follows from condition (1)).

Case II. $\beta_2' \neq 0$: Then the function $\beta\colon \mathbb{R} \setminus \{-\beta_2/\beta_2'\} \to \mathbb{R}$ defined by

$$\beta(\lambda) := \frac{\beta_1 + \beta_1'\lambda}{\beta_2 + \beta_2'\lambda}$$

is strictly increasing in the interval $(-\infty, -\beta_2/\beta_2')$ and in the interval $(-\beta_2/\beta_2', \infty)$ (again this follows from condition (1)).

Case II(a). $\lambda_k^{(1)} < -\beta_2/\beta_2'$. Then from Theorem 1 it follows that $y_k^{(1)}$ has exactly $(k-1)$ zeros in (x_0, x_n) and from Theorem 1 and Theorem 2.1 in [3] it follows that $y_k^{(2)}$ also has exactly $(k-1)$ zeros in (x_0, x_n). Thus by Theorem 1 $\lambda_k^{(2)} < -\beta_2/\beta_2'$. Again from Theorem 2.1 in [3] it follows that $y_k^{(1)}(x_n) \neq 0$. The assumption $y_k^{(2)}(x_n) = 0$ implies from the boundary condition (5) that $\lambda_k^{(2)} = -\beta_2/\beta_2'$ which is a contradiction to the formula just derived. The assumption $y_k^{(2)} \neq 0$ implies by again using Theorem 2.1 in [3] that $(r_1(y_k^{(1)})'/y_k^{(1)})(x_n) > (r_2(y_k^{(2)})'/y_k^{(2)}(x_n)$. With (5) the latter is equivalent to $\beta(\lambda_k^{(1)}) > \beta(\lambda_k^{(2)})$. This contradicts (14) and the strictly increasing of the function β in the interval $(-\infty, -\beta_2/\beta_2')$.

Case II(b). $\lambda_k^{(1)} \geqslant -\beta_2/\beta_2'$: Then from Theorem 1 it follows that $y_k^{(1)}$ has exactly $(k-2)$ zeros in (x_0, x_n). Furthermore from (14) and Theorem 1 it follows that $y_k^{(2)}$ also has exactly $(k-2)$ zeros in (x_0, x_n). Now we can proceed as in Case II(a) where we now have the strictly increasing of β in the interval $(-\beta_2/\beta_2', \infty)$ to get a contradiction. ∎

Remark. (a) The same condition (13) as in the case of Sturm–Liouville eigenvalue problems with λ-independent boundary conditions gives the validity of formula (12) (see [1, 3]).

(b) The examples $a = 0$, $b = \pi$, $r_j = 1$, $q_j = -10$, $p_j = j$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_1' = 1$, $\beta_2 = 1$, $\beta_2' = 0$, for $j = 1, 2$ show that without the condition (13) Theorem 2 would be false (compare also [1]).

(c) As in [1, 3] condition (13) is satisfied if at least one of the following two conditions is fulfilled:

(i) $p_1 = p_2$ (as in all problems in Liouville normal form)

(ii) $\lambda_k^{(1)} \geqslant 0$.

(d) $\lambda_k^{(2)} \geqslant 0$ is a further sufficient condition for (12). This can be proved in a similar way.

We introduce some further notation. For any $f \in PC_\pi^0[a, b]$ define step functions $f^+, f^-, \tilde{f} \in PC_\pi^0[a, b]$ by

$$f^+(x) := \sup_{(x_{i-1}, x_i)} f, \; f^-(x) := \inf_{(x_{i-1}, x_i)} f, \; \tilde{f} := f(x_{i-1/2})$$

when $x_{i-1} < x \leqslant x_i$, where $x_{i-1/2} := \frac{1}{2}(x_{i-1} + x_i)$, $i = 1, \ldots, n$. Denote by

$(2^+), ..., (5^+)$, $(2^-), ..., (5^-)$ and $(\tilde{2}), ..., (\tilde{5})$ the eigenvalue problems obtained from (2), ..., (5) by replacing p, q, r by $p^+, q^+, r^+, p^-, q^-, r^-$ and $\tilde{p}, \tilde{q}, \tilde{r}$, respectively. For $k = 1, 2, ...$ let $\lambda_k^{\pm}$, $\tilde{\lambda}_k$ denote the kth eigenvalue of $(2^{\pm}), ..., (5^{\pm})$ and $(\tilde{2}), ..., (\tilde{5})$, respectively. In the following we always assume that at least one of the following conditions is satisfied for the eigenvalue problem (2), ..., (5): $p = 1$ or $\lambda_k \geqslant 0$. Then Theorem 2 shows that

$$\lambda_k^- \leqslant \lambda_k \leqslant \lambda_k^+ .$$

Using calculations analogous to those used in [3] the bounds $\lambda_k^{\pm}$ for λ_k can be determined by a suitable modification of the algorithm described in [3].

Eigenvalue problems of the type considered above arise in many physical and technical problems; see [5, 6, 8–10] for examples.

3. Convergence of the Method

In this section we assume that $p, q, r \in C^1[a, b]$ and that the partition π is uniform $(x_i - x_{i-1} = 1/n(b-a) = h)$. We now prove that the approximate eigenvalues $\lambda_k^{\pm}$ represent $O(h)$ approximations to λ_k. The analysis is similar to that used in [10]. Therefore we describe the proofs rather briefly and only the most necessary steps from them are given. Furthermore the two cases of the eigenvalue problems $(2^+), ..., (5^+)$, $(2^-), ..., (5^-)$, respectively, are treated simultaneously. We formulate with the necessary modifications some lemmas analogous to Lemmas 1–4 and Theorem 4 in [10].

Lemma 1. *If $f \in C^1[a, b]$, then*

$$\|f - f^{\pm}\|_\infty \leqslant h\|f'\|_\infty,$$

$$\|f^{\pm}\|_\infty \leqslant \|f\|_\infty + h\|f'\|_\infty .$$

If $f \geqslant f_0$ (const) > 0, then

$$\left\|\frac{1}{f^{\pm}}\right\|_\infty \leqslant \left\|\frac{1}{f}\right\|_\infty + h\left\|\left(\frac{1}{f}\right)'\right\|_\infty .$$

($\|\cdot\|$ *denotes the* sup-norm *on* $[a, b]$).

Denote by $y_\mu^{\pm}$ the unique solution of the initial value problem

$$(r^{\pm}(y_\mu^{\pm})')' + (\mu p^{\pm} - q^{\pm})\, y_\mu^{\pm} = 0, \tag{15}$$

$$y_\mu^{\pm}(a) = -\alpha_2, \qquad (r^{\pm}(y_\mu^{\pm})')(a) = -\alpha_1, \tag{16}$$

which, for fixed h is allowed to depend on the real parameter μ for

$$\mu \in S_{h,k} := \{\mu \in \mathbb{R} \mid |\mu - \lambda_k| \leqslant h^{1-\varepsilon}\}$$

for any fixed $\varepsilon \in (0, 1)$. The initial value problem (15), (16) is equivalent to the integral equation

$$y_\mu^\pm(x) = -\alpha_2 - \alpha_1 \int_a^x \frac{1}{r^\pm(s)}\, ds - \mu \int_a^x \frac{1}{r^\pm(t)} \int_a^t p^\pm(s)\, y_\mu^\pm(s)\, ds\, dt$$
$$+ \int_a^x \frac{1}{r^\pm(t)} \int_a^t q^\pm(s)\, y_\mu^\pm(s)\, ds\, dt. \tag{17}$$

Then the following Lemma can be proved similar to Lemma 2 in [10].

LEMMA 2. *For $\mu \in S_{h,k}$ and $h < 1$*

$$\|y_\mu^\pm\|_\infty \leqslant \tilde{M} \cosh M(b-a),$$

where

$$M^2 = \left(\left\|\frac{1}{r}\right\|_\infty + \left\|\left(\frac{1}{r}\right)'\right\|_\infty\right)$$
$$\times ((1 + |\lambda_k|)(\|p\|_\infty + \|p'\|_\infty) + \|q\|_\infty + \|q'\|_\infty)$$

and

$$\tilde{M} = |\alpha_2| + |\alpha_1|(b-a)\left(\left\|\frac{1}{r}\right\|_\infty + \left\|\left(\frac{1}{r}\right)'\right\|_\infty\right).$$

Moreover

$$\|(y_\mu^\pm)'\|_\infty \leqslant |\alpha_1| \left(\left\|\frac{1}{r}\right\|_\infty + \left\|\left(\frac{1}{r}\right)'\right\|_\infty\right) + (b-a)\, M^2 \tilde{M} \cosh M(b-a).$$

Proof. We have

$$\left|\frac{1}{r^\pm(t)} \int_a^t p^\pm(s)\, y_\mu^\pm(s)\, ds\right| \leqslant \left\|\frac{1}{r^\pm}\right\|_\infty \|p^\pm\|_\infty \int_a^t |y_\mu^\pm(s)|\, ds \tag{18}$$

and

$$\left|\frac{1}{r^\pm(t)} \int_a^t q^\pm(s)\, y_\mu^\pm(s)\, ds\right| \leqslant \left\|\frac{1}{r^\pm}\right\|_\infty \|q^\pm\|_\infty \int_a^t |y_\mu^\pm(s)|\, ds. \tag{19}$$

Using this, estimating in (17), and applying a variant of the Gronwall inequality the bound on $\|y_\mu^\pm\|_\infty$ follows from Lemma 1 and the definition of $S_{h,k}$. The bound on $\|(y_\mu^\pm)'\|_\infty$ follows by doing only one integration in (15), estimating in a similar way and applying the bound for $\|y_\mu^\pm\|_\infty$. ∎

Similar to Lemma 3 in [10] the following Lemma can be proved.

LEMMA 3. *There exists a constant C independent of h such that for all $\mu_1, \mu_2 \in S_{h,k}$ we have*

$$\| y^{\pm}_{\mu_1} - y^{\pm}_{\mu_2} \|_\infty \leqslant C|\mu_1 - \mu_2|$$

and

$$\| r^{\pm}(y^{\pm}_{\mu_1})' - r^{\pm}(y^{\pm}_{\mu_2})' \|_\infty \leqslant C|\mu_1 - \mu_2|.$$

Proof. We have from (17)

$$(y^{\pm}_{\mu_1} - y^{\pm}_{\mu_2})(x) = (\mu_2 - \mu_1) \int_a^x \frac{1}{r^{\pm}(t)} \int_a^t p^{\pm} y^{\pm}_{\mu_1}(s)\, ds\, dt$$

$$+ \int_a^x \frac{1}{r^{\pm}(t)} \int_a^t (\mu_2 p^{\pm}(s) - q^{\pm}(s))(y^{\pm}_{\mu_2}(s) - y^{\pm}_{\mu_1}(s))\, ds\, dt.$$

Estimating in this equation, making use of (18), (19), and applying the same variant of the Gronwall inequality as before we get the first inequality. The second inequality follows by doing only one integration in (15) and applying the first inequality just proved. It follows from Lemmas 1 and 2 that the constant C can be chosen independently of h. ∎

Let ϕ_k be the eigenfunction corresponding to the eigenvalue λ_k of the eigenvalue problem (2), ..., (5) normalized according to

$$\phi_k(a) = -\alpha_2, \; (r\phi_k')(a) = -\alpha_1. \tag{20}$$

Then similar to Lemma 4 in [10] the following Lemma can be proved.

LEMMA 4. *For all $\mu \in S_{h,k}$ we have*

$$\| \phi_k - y^{\pm}_{\mu} \|_\infty = O(h^{1-\varepsilon}) \tag{21}$$

and

$$\| r\phi_k' - r^{\pm}(y^{\pm}_{\mu})' \|_\infty = O(h^{1-\varepsilon}) \tag{22}$$

for h sufficiently small.

Proof. The initial value problem (2), (20) for the kth eigenfunction ϕ_k is equivalent to the integral equation

$$\phi_k(x) = -\alpha_2 - \alpha_1 \int_a^x \frac{1}{r(s)}\, ds - \lambda_k \int_a^x \frac{1}{r(t)} \int_a^t p(s)\, \phi_k(s)\, ds\, dt$$

$$+ \int_a^x \frac{1}{r(t)} \int_a^t q(s)\, \phi_k(s)\, ds\, dt. \tag{23}$$

Making use of this and (17) it follows that

$$(y_\mu^\pm - \phi_k)(x) = (\lambda_k - \mu) \int_a^x \frac{1}{r^\pm} \int_a^t p^\pm y_\mu^\pm \, ds\, dt + \alpha_1 \int_a^x \left(\frac{1}{r} - \frac{1}{r^\pm}\right) dt$$
$$+ \lambda_k \int_a^x \frac{1}{r^\pm} \int_a^t p^\pm (\phi_k - y_\mu^\pm)\, ds\, dt$$
$$- \int_a^x \frac{1}{r^\pm} \int_a^t q^\pm (\phi_k - y_\mu^\pm)\, ds\, dt$$
$$- \lambda_k \int_a^x \frac{1}{r^\pm} \int_a^t p^\pm \phi_k \, ds\, dt + \lambda_k \int_a^x \frac{1}{r} \int_a^t p\phi_k \, ds\, dt$$
$$+ \int_a^x \frac{1}{r^\pm} \int_a^t q^\pm \phi_k \, ds\, dt - \int_a^x \frac{1}{r} \int_a^t q\phi_k \, ds\, dt. \tag{24}$$

From this we obtain, by applying Lemma 2 and the Gronwall inequality,

$$|(y_\mu^\pm - \phi_k)(x)| \leqslant \left(|\lambda_k - \mu| \left\| \frac{1}{r^\pm} \right\|_\infty \|p^\pm\|_\infty \tilde{M} \frac{1}{2} (b-a)^2 \cosh M(b-a) \right.$$
$$\left. + \left| \int_a^b G_{h,k}(t)\, dt \right| \right) \cosh M(b-a) \tag{25}$$

for all $x \in [a, b]$, where

$$\left| \int_a^b G_{h,k}(t)\, dt \right| := \left| \int_a^b \left(\alpha_1 \frac{r^\pm - r}{r^\pm r} - \frac{1}{r^\pm} \int_a^t (\lambda_k p^\pm - q^\pm)\, \phi_k \, ds \right.\right.$$
$$\left.\left. + \frac{1}{r} \int_a^t (\lambda_k p - q)\, \phi_k \, ds \right) dt \right|$$
$$\leqslant |\alpha_1| \, \|r^\pm - r\|_\infty \int_a^b \frac{1}{r^\pm r}\, dt$$
$$+ \|r^\pm - r\|_\infty \, |\lambda_k| \int_a^b \frac{1}{r^\pm r} \int_a^t p|\phi_k| \, ds\, dt$$
$$+ \|r - r^\pm\|_\infty \int_a^b \frac{1}{r^\pm r} \int_a^t |q| \, |\phi_k| \, ds\, dt$$
$$+ \|q^\pm - q\|_\infty \int_a^b \frac{1}{r^\pm} \int_a^t |\phi_k| \, ds\, dt$$
$$+ \|p - p^\pm\|_\infty \, |\lambda_k| \int_a^b \frac{1}{r^\pm} \int_a^t |\phi_k| \, ds\, dt. \tag{26}$$

The result (21) now follows as a consequence of Lemma 1 and the definition of $S_{h,k}$ since the integrals appearing in (26) are bounded independently of h for h sufficiently small. The result (22) follows by doing only one integration in (15), (16), and in (2), (20), and then applying (21). ∎

In the following we use the notation

$$R'_b(y) := \beta'_1 y(b) - \beta'_2(ry')(b),$$
$$R'_{b,h}(y^\pm) := \beta'_1 y^\pm(b) - \beta'_2(r^\pm(y^\pm)')(b).$$

Let $\phi_k^\pm$ denote the eigenfunction corresponding to the eigenvalue $\lambda_k^\pm$ normalized according to

$$\phi_k^\pm(a) = -\alpha_2, (r^\pm(\phi_k^\pm)')(a) = -\alpha_1 \tag{27}$$

and let $A^\pm$ the self-adjoint operator in the Hilbertspace $(H^\pm, [\cdot,\cdot]^\pm)$ corresponding to the eigenvalue problem $(2^\pm)$, ..., $(5^\pm)$. Then it follows that

$$[A(\phi_k, R'_b(\phi_k)), (\phi_k^\pm, R'_{b,h}(\phi_k^\pm))] - [A^\pm(\phi_k^\pm, R'_{b,h}(\phi_k^\pm)), (\phi_k, R'_b(\phi_k))]^\pm$$
$$= \int_a^b (r - r^\pm)\, \phi'_k(\phi_k^\pm)'\, dt + \int_a^b (q - q^\pm)\, \phi_k \phi_k^\pm\, dt \tag{28}$$

and furthermore

$$\lambda_k - \lambda_k^\pm = \frac{\left(\begin{array}{l}\int_a^b (r - r^\pm)\, \phi'_k(\phi_k^\pm)'\, dt \\ \qquad + \int_a^b (q - q^\pm)\, \phi_k \phi_k^\pm\, dt - \lambda_k \int_a^b (p - p^\pm)\, \phi_k \phi_k^\pm\, dt\end{array}\right)}{\int_a^b p^\pm \phi_k \phi_k^\pm\, dt + (1/\rho)\, R'_b(\phi_k)\, R'_{b,h}(\phi_k^\pm)}. \tag{29}$$

Equation (29) suggests to define a mapping $T^\pm : S_{h,k} \to \mathbb{R}$ by

$$T^\pm(\mu) := \lambda_k - \frac{\left(\begin{array}{l}\int_a^b (r - r^\pm)\, \phi'_k(y_\mu^\pm)'\, dt \\ \qquad + \int_a^b (q - q^\pm)\, \phi_k y_\mu^\pm\, dt - \lambda_k \int_a^b (p - p^\pm)\, \phi_k y_\mu^\pm\, dt\end{array}\right)}{\int_a^b p^\pm \phi_k y_\mu^\pm\, dt + (1/\rho) R'_b(\phi_k) R'_{b,h}(y_\mu^\pm)}, \tag{30}$$

where $y_\mu^\pm$ is the solution of (15), (16). Now similar to Theorem 4 in [10] the following Lemma can be proved.

LEMMA 5. *For h sufficiently small the mapping $T^\pm$ is a contraction from $S_{h,k}$ into $S_{h,k}$.*

Proof. For example, we only prove that $T^\pm$ maps $S_{h,k}$ into $S_{h,k}$ for h sufficiently small. Further steps of the proof can be done analogously to that of Theorem 4 in [10] and we omit them. Let $\mu \in S_{h,k}$; then it follows from (30) that the numerator of $|T^{\pm}(\mu)-\lambda_k|$ can be estimated:

$$(\|r-r^{\pm}\|_{\infty}\,\|\phi_k'\|_{\infty}\,\|(y_{\mu}^{\pm})'\|_{\infty}+\|q-q^{\pm}\|_{\infty}\,\|\phi_k\|_{\infty}\,\|y_{\mu}^{\pm}\|_{\infty}$$
$$+|\lambda_k|\,\|p-p^{\pm}\|_{\infty}\,\|\phi_k\|_{\infty}\,\|y_{\mu}^{\pm}\|_{\infty})(b-a).$$

Thus the numerator is $O(h)$ from Lemmas 1 and 2. For the first term in the denominator we have

$$\int_a^b p^{\pm}\phi_k\,y_{\mu}^{\pm}\,dt=\int_a^b p\phi_k^2\,dt+O(h^{1-\varepsilon})$$

from Lemmas 1 and 4. For the second term of the denominator we have

$$\frac{1}{\rho}R_b'(\phi_k)\,R_{b,h}'(y_{\mu}^{\pm})=\frac{1}{\rho}(R_b'(\phi_k))^2+O(h^{1-\varepsilon})$$

by applying Lemma 4. Thus $|T^{\pm}(\mu)-\lambda_k|=O(h)$ as $h\to 0$. This implies that for h sufficiently small $T^{\pm}$ maps $S_{h,k}$ into $S_{h,k}$. ∎

We are now ready to prove

THEOREM 3. *For h sufficiently small, there exist constants C_1, C_2, C_3 independent of h such that*

$$|\lambda_k^{\pm}|\leqslant C_1,\ \|\phi_k^{\pm}\|_{\infty}\leqslant C_2,\ \|r^{\pm}(\phi_k^{\pm})'\|_{\infty}\leqslant C_3. \tag{31}$$

Moreover,

$$|\lambda_k-\lambda_k^{\pm}|=O(h), \tag{32}$$

$$\|\phi_k-\phi_k^{\pm}\|_{\infty}=O(h), \tag{33}$$

$$\|r\phi_k'-r^{\pm}(\phi_k^{\pm})'\|_{\infty}=O(h). \tag{34}$$

Proof. We choose h sufficiently small so that $T^{\pm}$ is a contraction on $S_{h,k}$ as in Lemma 5. Then it follows from (29) and (30) that $\lambda_k^{\pm}$ is the fixed point of $T^{\pm}$. Furthermore from (15), (16), and (27) we have $\phi_k^{\pm}=y_{\lambda_k^{\pm}}^{\pm}$. Then Lemma 4 implies

$$\|\phi_k-\phi_k^{\pm}\|_{\infty}=O(h^{1-\varepsilon}),\qquad \|r\phi_k'-r^{\pm}(\phi_k^{\pm})'\|_{\infty}=O(h^{1-\varepsilon})$$

and since $\lambda_k^{\pm}\in S_{h,k}$ we also have

$$|\lambda_k-\lambda_k^{\pm}|=O(h^{1-\varepsilon}).$$

From this formula (31) follows. The error estimation (32) is obtained by estimating (29) by application of (31) and Lemma 1. The error estimation (33) follows by application of (31), (32), Lemma 1 to (25), (26) for $\mu = \lambda_k$. An analogous argument proves (34). ∎

In the case of Theorem 3, $O(h^2)$ rates of convergence cannot be obtained in general. But this can be proved for the approximate eigenvalues, eigenfunctions, and its quasiderivatives of the eigenvalue problem $(\tilde{2})$, ..., $(\tilde{5})$.

THEOREM 4. *Let $p, q, r \in C^3[a, b]$. Then for h sufficiently small, there exist constants $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ independent of h such that*

$$|\tilde{\lambda}_k| \leqslant \tilde{C}_1, \; \|\tilde{\phi}_k\|_\infty \leqslant \tilde{C}_2, \; \|\tilde{r}\tilde{\phi}'_k\|_\infty \leqslant \tilde{C}_3. \tag{35}$$

Moreover

$$|\lambda_k - \tilde{\lambda}_k| = O(h^2), \tag{36}$$

$$\|\phi_k - \tilde{\phi}_k\|_\infty = O(h^2), \tag{37}$$

$$\|r\phi'_k - \tilde{r}\tilde{\phi}'_k\|_\infty = O(h^2). \tag{38}$$

Proof. Analogous to our Theorem 3 and to Corollary 4.1 of [10] the four estimations (35)–(38) can be proved since now Lemma 2 of [15] is applicable. Therefore we omit further details of the proof. ∎

The eigenvalues $\tilde{\lambda}_k$ must be computed by an algorithm analogous to that which is used in the computation of the $\lambda_k^{\pm}$. But to a certain extent without further computations we can get $O(h^2)$-approximations to the λ_k by the arithmetic means $\frac{1}{2}(\lambda_k^- + \lambda_k^+)$.

THEOREM 5. *Let $p, q, r \in C^3[a, b]$. Then for h sufficiently small*

$$|\lambda_k - (1/2)(\lambda_k^+ + \lambda_k^-)| = O(h^2).$$

Proof. With Theorems 4 and 3 the conclusion follows similar to Theorem 1 in [13], if the Rayleigh quotients considered in [13] are replaced by Rayleigh quotients defined about the quadratic form J considered in Section 2. We omit further details of the proof. ∎

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REFERENCES

1. A. L. ANDREW, Computation of higher Sturm–Liouville eigenvalues, *in* "Proceedings, 11th Manitoba Conference on Numerical Mathematics and Computing 1981," Vol. 34, pp. 3–16, Congressus Numerantium, 1982.
2. A. L. ANDREW, Numerical solution of eigenvalue problems for ordinary differential equations, *in* "Computational Techniques and Applications: CT AC-83" (J. Noyce and C. Fletcher, Eds.), pp. 841–852, North-Holland, Amsterdam, 1984.
3. A. L. ANDREW, F. R. DE HOOG, AND P. J. ROBB, Leighton's bounds for Sturm–Liouville eigenvalues, *J. Math. Anal. Appl.* **83** (1981), 11–19.
4. A. I. BENEDEK AND R. PANZONE, On Sturm–Liouville problems with the squareroot of the eigenvalue parameter contained in the boundary conditions, *Notas Algebra Anal.*, No. 10, (1981).
5. W. E. BOYCE AND G. H. HANDELMAN, Vibration of rotating beams with tip mass, *Z. Angew. Math. Phys.* **12** (1961), 369–392.
6. L. COLLATZ, "Eigenwertaufgaben mit technischen Andwendungen," 2nd ed., Leipzig, Akad. Verlagsgesellschaft, 1963.
7. W. B. DAY, More bounds for eigenvalues, *J. Math. Anal. Appl.* **46** (1974), 523–532.
8. C. T. FULTON, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh* **77**A (1977), 293–308.
9. C. T. FULTON, Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A* **87** (1980), 1–34.
10. C. T. FULTON AND S. PRUESS, Numerical methods for a singular eigenvalue problem with eigenparameter in the boundary conditions, *J. Math. Anal. Appl.* **71** (1979), 431–462.
11. D. B. HINTON, An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition, *Quart. J. Math. Oxford Ser.* (2) **30** (1979), 33–42.
12. W. LEIGHTON, Upper and lower bounds for eigenvalues, *J. Math. Anal. Appl.* **35** (1971), 381–388.
13. J. W. PAINE AND A. L. ANDREW, Bounds and higher-order estimates for Sturm–Liouville eigenvalues, *J. Math. Anal. Appl.* **96** (1983), 388–394.
14. S. PRUESS, Estimating the eigenvalues of Sturm–Liouville problems by approximating the differential equation, *SIAM J. Numer. Anal.* **10** (1973), 55–68.
15. S. PRUESS, High order approximations to Sturm–Liouville eigenvalues, *Numer. Math.* **24** (1975), 241–247.
16. M. OCHMANN, "Eigenwertprobleme mit linear in beiden Randbedingungen auftretendem Eigenwertparameter—eine Verallgemeinerung der Fultonschen Methode," Diplomarbeit, Hagen, 1986.
17. J. WALTER, Regular eigenvalue problems with eigenvalue parameter in the boundary conditions, *Math. Z.* **133** (1973), 301–312.